Lecture 1 - Frequency domain processing of signals and images

Organization

- Lecture: Monday, 10:15 - 11:45
- Exercise: Wednesday, 08:15 - 09:45
- Moodle page
- To enroll, please send an email from your FSU account with the subject "moodle IP II"; I will then enroll you

Outline

- Recap: Fourier series, discrete and fast Fourier transform
- Filtering in the time and spatial domain
- Filtering in the frequency domain
- Convolution (definition and convolution theorem)
- Correlation and auto-correlation

Fourier Series expansion

Under mild regularity conditions, we can represent $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{1}$$

with

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\,. \tag{2}$$

In complex form, we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3}$$

with

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\,. \tag{4}$$

These relations also hold for multi-dimensional signal such as images.

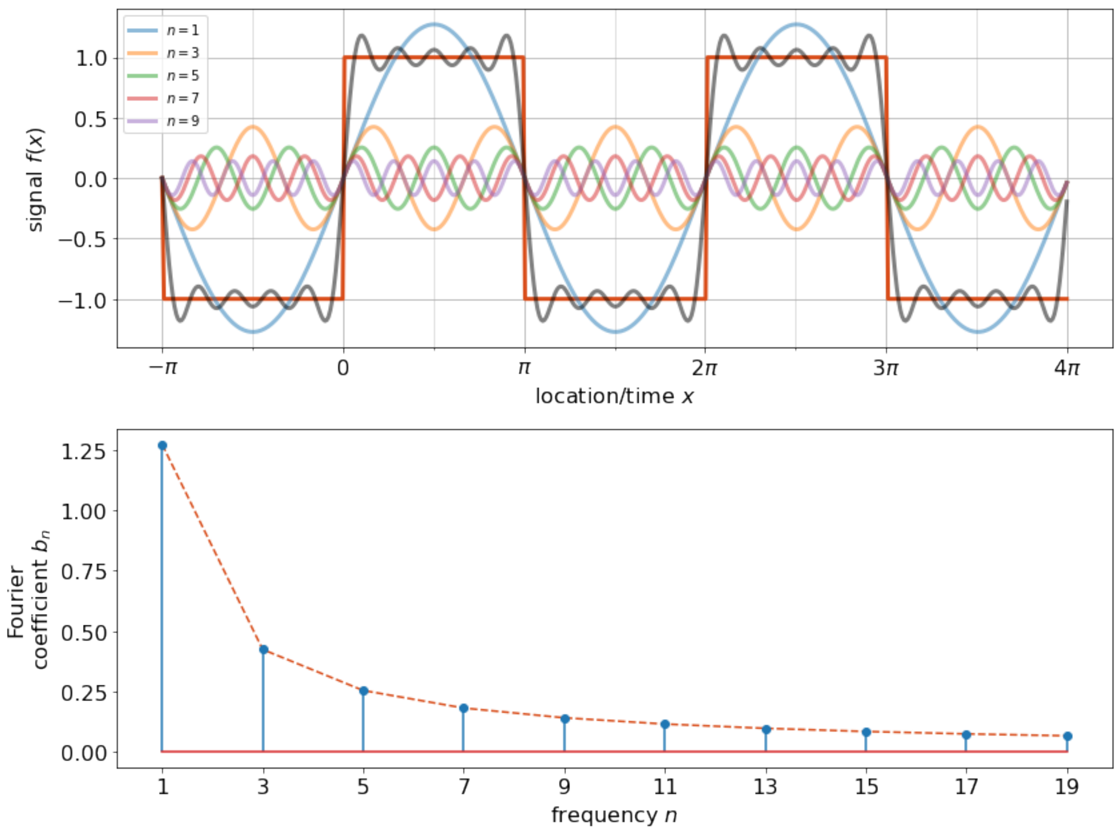
Example: square wave

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin(nx)}{n}$$

[1]:
```
# Fourier transforms of signals and images
import numpy as np
import matplotlib.pylab as plt
import scipy.signal as signal

colors = [(.85, .3, .1), (.15, .35, .6), (.95, .7, 0.1), (.0, .0, .0),
          (.8, .8, .8)]

def square_wave(x):
    increment = (x[1]-x[0]) * 1e-5
    return 0.5 * (signal.square(x+increment) + signal.square(x-increment))

def s(x, n):
    """Contribution to the Fourier expansion of the square wave. """
    return 4/np.pi * np.sin(n*x) / n
```

[2]:
```
N = 512
x = np.linspace(-np.pi, 4*np.pi, N, endpoint=False)

plt.rc('font', size=16)
fig, axes = plt.subplots(2, 1, figsize=(12, 9))
ax = axes[0]
ax.plot(x, square_wave(x), color=colors[0], lw=3)
ax.set_xticks(np.linspace(x.min(), x.max(), 6))
ax.set_xticks(np.linspace(x.min(), x.max(), 6)+0.5*np.pi, minor=True)
ax.grid(True, which='minor', alpha=0.5)
ax.grid(True, which='major')
ax.set_xticklabels([r'$-\pi$', r'$0$', r'$\pi$', r'$2\pi$', r'$3\pi$',␣
 ↪r'$4\pi$'])

s_sum = 0.
for n in (1, 3, 5, 7, 9):
    s_n = s(x, n)
    ax.plot(x, s_n, lw=3, alpha=0.5, label=r'$n={}$'.format(n))
    s_sum += s_n
ax.plot(x, s_sum, color=colors[3], lw=3, alpha=0.5)
ax.set_xlabel(r'location/time $x$')
ax.set_ylabel(r'signal $f(x)$')
ax.legend(loc=2, fontsize=10);

#Fourier coefficients
ax = axes[1]
```

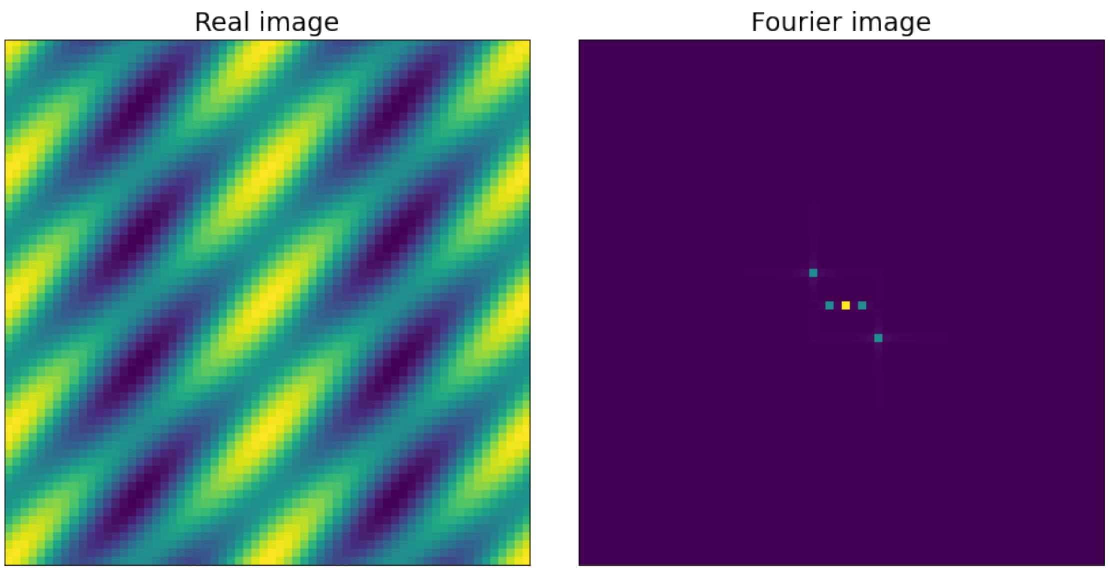
```
N_max = 19
n = np.arange(1, N_max+1, 2)
b = 4/(np.pi * n)
S = np.sin(np.multiply.outer(x, n))

# plot Fourier coefficient b_n
ax.plot(n, b, ls='--', color=colors[0])
ax.stem(n, b, use_line_collection=True)
ax.set_xlabel(r'frequency $n$')
ax.set_ylabel('Fourier\n'+r'coefficient $b_n$')
ax.set_xticks(n);
fig.tight_layout()
```



We can do the same with images (two-dimensional signals $f(x, y)$):

```
[3]: def ft(f):
         """Two-dimensional Fourier transform with zero-frequency shifted
         to image center. """
         return np.fft.fftshift(np.fft.fft2(f))

     x = y = np.linspace(-np.pi, np.pi, 64)
```

```python
X, Y = np.meshgrid(x, y)
f = 1 + np.cos(4*(X+Y)) + np.cos(2*X)

subplot_kw = dict(xticks=[], yticks=[])
fig, ax = plt.subplots(1, 2, figsize=(12, 6), subplot_kw=subplot_kw)
ax[0].set_title('Real image')
ax[0].imshow(f, interpolation=None)
ax[1].set_title('Fourier image')
ax[1].imshow(np.abs(ft(f)), interpolation=None)
fig.tight_layout()
```



Discrete Fourier transform

The *discrete Fourier transform* (DFT) is the finite sum version of the integral (Eq. 4):

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \quad \Longrightarrow \quad c_n = \frac{1}{N}\sum_{m=0}^{N-1} f_m\, e^{-2\pi i n m/N} \tag{5}$$

where N is a finite integer. The signal is evaluated ("sampled") at discrete locations/times $x_m = 2\pi m/N$ where $m \in \{0, 1, \ldots, (N-1)\}$ such that $f_m = f(x_m) = f(2\pi m/N)$. By convention, we rather work with

$$\hat{f}_n = N c_n = \sum_{m=0}^{N-1} f_m\, e^{-2\pi i n m/N}$$

where $\hat{f}_n$ is the Fourier transform of f_n.

The DFT can be written compactly using matrix-vector multiplication. If $\boldsymbol{f} \in \mathbb{C}^N$ is the N-dimensional vector/array $\boldsymbol{f} = [f_0, \ldots, f_{N-1}]$ then

$$\hat{\boldsymbol{f}} = \boldsymbol{F}_N \boldsymbol{f} \tag{6}$$

where $\boldsymbol{F}_N$ is the N-dimensional DFT matrix with elements $(\boldsymbol{F}_N)_{mn} = \omega^{mn}$ where $\omega = e^{-2\pi i/N}$ is the N-th root of unity. In practice, it is much faster to compute the DFT by using the Fast Fourier Transform (FFT).

The DFT can be inverted:

$$f_m = \frac{1}{N} \sum_{n=0}^{N-1} \hat{f}_n \, e^{2\pi i n m/N}$$

where $\hat{f}_n$ is the Fourier transform of f_n. That is, the inverse DFT matrix is

$$(\boldsymbol{F}_N^{-1})_{nm} = \frac{1}{N} e^{2\pi i n m/N}$$

Let's write out the DFT and inverse DFT matrix explicitly:

$$\begin{aligned}
\boldsymbol{F}_N &= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{pmatrix} \\
\boldsymbol{F}_N^{-1} &= \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^* & (\omega^*)^2 & \cdots & (\omega^*)^{N-1} \\ 1 & (\omega^*)^2 & (\omega^*)^4 & \cdots & (\omega^*)^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (\omega^*)^{N-1} & (\omega^*)^{2(N-1)} & \cdots & (\omega^*)^{(N-1)^2} \end{pmatrix} = \frac{1}{N} \boldsymbol{F}_N^* .
\end{aligned} \tag{7}$$

Example: DFT for $N = 2$

The signal has two elements $\boldsymbol{f} = [f_0, f_1]$. The square root of unity is

$$\omega = \exp\{-2\pi i/N\} \overset{N=2}{=} \exp\{-\pi i\} = \cos\pi - i\sin\pi = -1$$

The DFT matrix is a 2×2 matrix:

$$\boldsymbol{F} = \begin{pmatrix} 1 & 1 \\ 1 & \omega \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Therefore,

$$\hat{\boldsymbol{f}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_0 + f_1 \\ f_0 - f_1 \end{pmatrix} .$$

[4]:
```python
# check equation for random signal
f = np.random.randn(2) + 1j * np.random.randn(2)
```

```
F = np.array([f[0]+f[1], f[0]-f[1]])
print(f'  Signal: {f}')
print(f'     FFT: {np.fft.fft(f)}')
print(f'Equation: {F}')
```

```
  Signal: [ 0.88719882+0.1454988j  -0.89724717+1.28676582j]
     FFT: [-0.01004834+1.43226462j  1.78444599-1.14126702j]
Equation: [-0.01004834+1.43226462j  1.78444599-1.14126702j]
```

DFT of two-dimensional signals

The DFT can also be defined for multi-dimensional signals such as images or volumes. For images $\boldsymbol{f} \in \mathbb{C}^{N\times N}$, for example, we have

$$\hat{f}_{kl} = \sum_{m=1}^{N} \sum_{n=1}^{N} \omega^{km} \omega^{ln} f_{mn} \tag{8}$$

In matrix-vector notation

$$\hat{\boldsymbol{f}} = \boldsymbol{F}_N \boldsymbol{f} \boldsymbol{F}_N \,. \tag{9}$$

The FFT implementation for 2D signals offered by NumPy is `np.fft.fft2`.

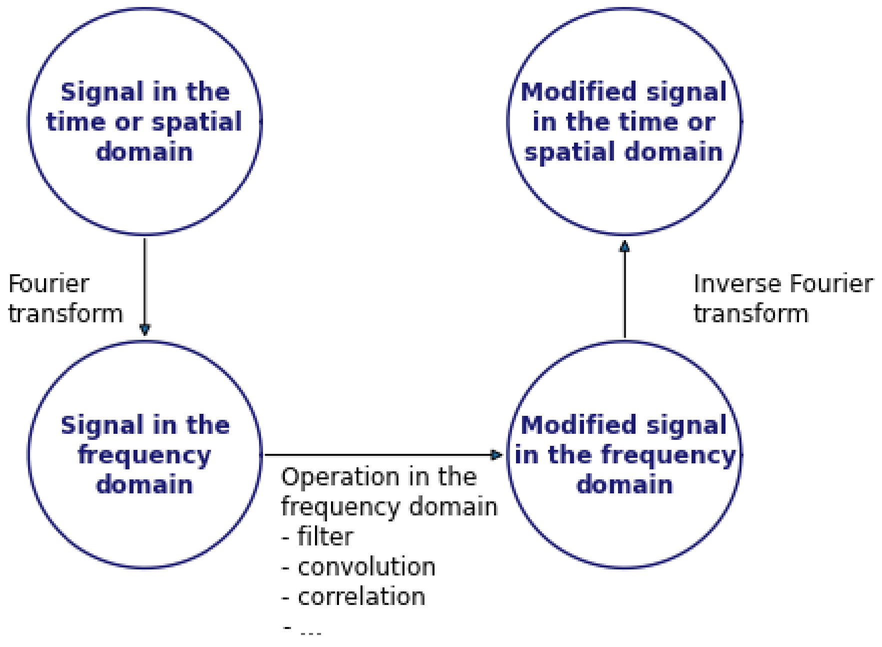
Fourier domain processing

Often it is convenient to process a signal / image in the frequency domain. To do so, we first compute the Fourier transform, apply some operation on the Fourier-transformed signal to obtain a modified signal in the frequency domain, and finally back-transform to the real domain by an inverse Fourier transform.

```
[5]: plt.rc('font', size=12)
fig, ax = plt.subplots(figsize=(8, 5))
ax.set_xlim(0, 16)
ax.set_ylim(-1, 9)
ax.axis('off')

centers = [(4, 7), (4, 2), (11, 2), (11, 7)]
radius = 1.7
texts = [
    "\n".join(["Signal in the", "time or spatial", "domain"]),
    "\n".join(["Signal in the", "frequency", "domain"]),
    "\n".join(["Modified signal", "in the frequency", "domain"]),
    "\n".join(["Modified signal", "in the time or", "spatial domain"]),
]

# Draw circles with text in the center
```

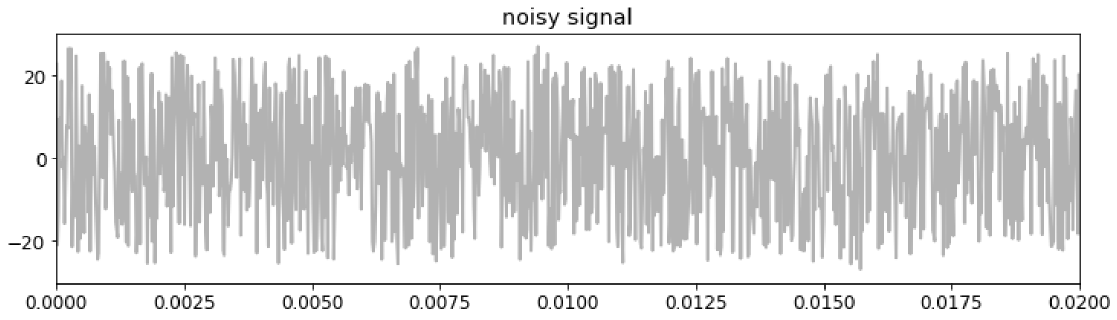
```python
for i, center in enumerate(centers):
    x, y = center
    theta = np.linspace(0, 2 * np.pi, 100)
    ax.plot(
        x + radius * np.cos(theta),
        y + radius * np.sin(theta),
        color="midnightblue",
    )
    ax.text(
        x, y,
        texts[i],
        horizontalalignment="center",
        verticalalignment="center",
        color="midnightblue",
        fontweight='bold',
    )

ax.annotate(
    "",
    (centers[1][0], centers[1][1] + radius),
    (centers[0][0], centers[0][1] - radius),
    arrowprops=dict(arrowstyle = "-|>"),
)
ax.annotate(
    "",
    (centers[2][0] - radius, centers[2][1]),
    (centers[1][0] + radius, centers[1][1]),
    arrowprops=dict(arrowstyle = "-|>"),
)
ax.annotate(
    "",
    (centers[3][0], centers[3][1] - radius),
    (centers[2][0], centers[2][1] + radius),
    arrowprops=dict(arrowstyle = "-|>"),
)

ax.annotate(
    "Fourier\ntransform",
    (centers[0][0] - 2, (centers[0][1] + centers[1][1])/2-0.5)
)


ax.annotate(
    "Inverse Fourier\ntransform",
    (centers[3][0] + 1, (centers[3][1] + centers[2][1])/2-0.5)
)
```

```
ax.annotate(
    "\n".join(["Operation in the\nfrequency domain",
               "- filter", "- convolution", "- correlation", r"- $\ldots$"]),
    ((centers[2][0] + centers[1][0])/2-1.5, (centers[2][1]-2.7))
)

fig.tight_layout()
```


Signal in the time or spatial domain
Modified signal in the time or spatial domain
Fourier transform
Inverse Fourier transform
Signal in the frequency domain
Modified signal in the frequency domain
Operation in the frequency domain
- filter
- convolution
- correlation
- ...

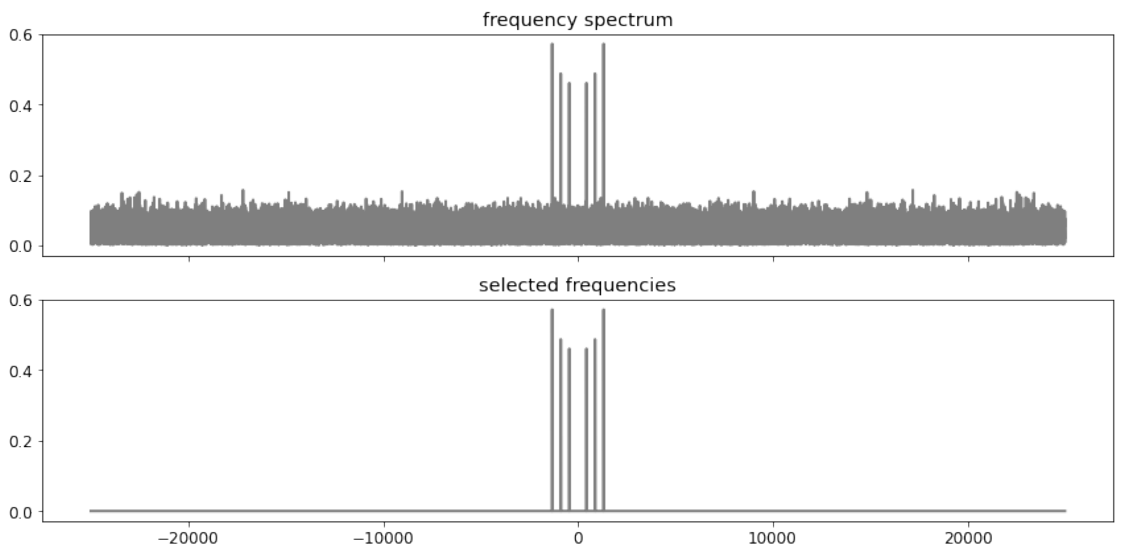
[6]:
```
"""
Example: filtering of a noisy signal
"""
# sampling rate in Hz
sampling_rate = 50000

# create a time axis
t = np.arange(0, 2, 1/sampling_rate)

# generate three signals
freqs = np.arange(1, 4) * 440
signals = np.sin(2*np.pi*np.multiply.outer(freqs, t))
signal = signals.sum(0)
xlim = (0., 0.02)

# generate noise and add it to the signal
noise = 25 * np.random.uniform(-1., 1., size=t.size)
```

```
noisy_signal = signal + noise

fig, ax = plt.subplots(figsize=(12, 3))
ax.plot(t, noisy_signal, color='k', ls='-', lw=2, alpha=0.3)
ax.set_title('noisy signal')
ax.set_xlim(*xlim);
```



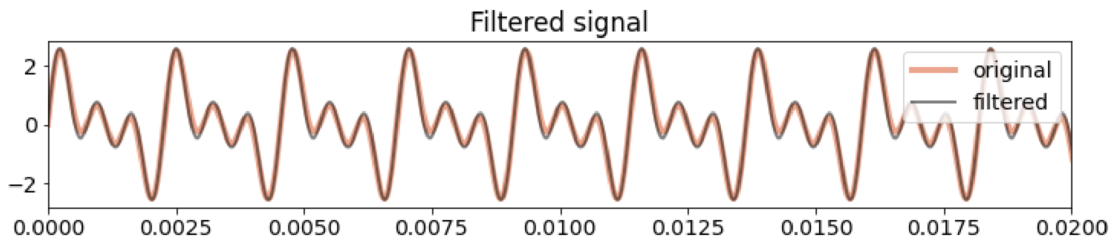
[7]:
```
# compute and show the frequency spectrum ---
ft = np.fft.fft(noisy_signal)
amplitudes = np.abs(ft)

# wrap amplitudes such that the zero frequency component is centered and scale
# the amplitudes by the number of data points
amplitudes = np.fft.fftshift(amplitudes) / t.size
frequencies = np.linspace(-0.5, 0.5, len(t)) * sampling_rate

# filter the signal: extract frequency components that exceed a threshold
mask = amplitudes > 0.5 * np.max(amplitudes)

# show the frequency spectrum
plot_kw = dict(color='k', lw=2, alpha=0.5)
fig, ax = plt.subplots(2, 1, figsize=(12, 6), sharex='all', sharey='all')
ax[0].plot(frequencies, amplitudes, **plot_kw)
ax[0].set_title('frequency spectrum')
ax[1].plot(frequencies, amplitudes*mask, **plot_kw)
ax[1].set_title('selected frequencies')
fig.tight_layout()
print('number of filtered elements:', mask.sum())
```

```
number of filtered elements: 6
```



```
[8]: # Transform the spectrum back to the time domain
     filtered = np.fft.ifft(ft * np.fft.ifftshift(mask)).real
```

```
[14]: plt.rc('font', size=14)
      fig, ax = plt.subplots(figsize=(12, 2))
      ax.plot(t, signal, lw=4, color=colors[0], alpha=0.5, label='original')
      ax.plot(t, filtered, label='filtered', **plot_kw)
      ax.set_title('Filtered signal')
      ax.legend(loc=1)
      ax.set_xlim(*xlim);
```



Convolution

An important operation on signals and images is a convolution. The convolution of a function f with another function g is defined as the integral of the product of the two functions after one is reversed and shifted:

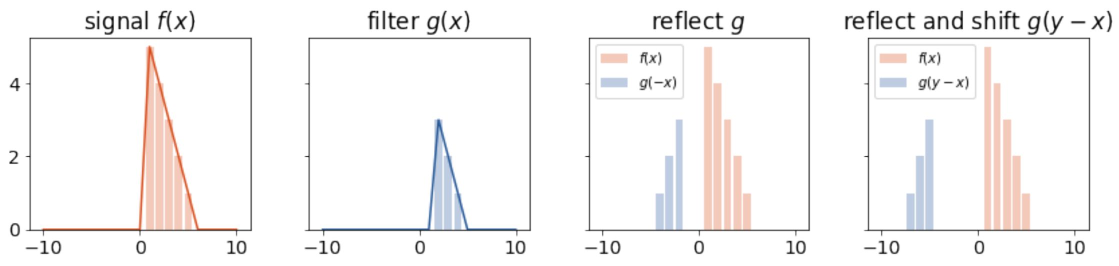
$$(f \otimes g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \tag{10}$$

Equation (10) can be interpreted as a weighted average of the function $f(y)$ at y where the weighting is given by $g(-y)$ simply shifted by x. As x changes, the weighting function emphasizes different parts of the input function.

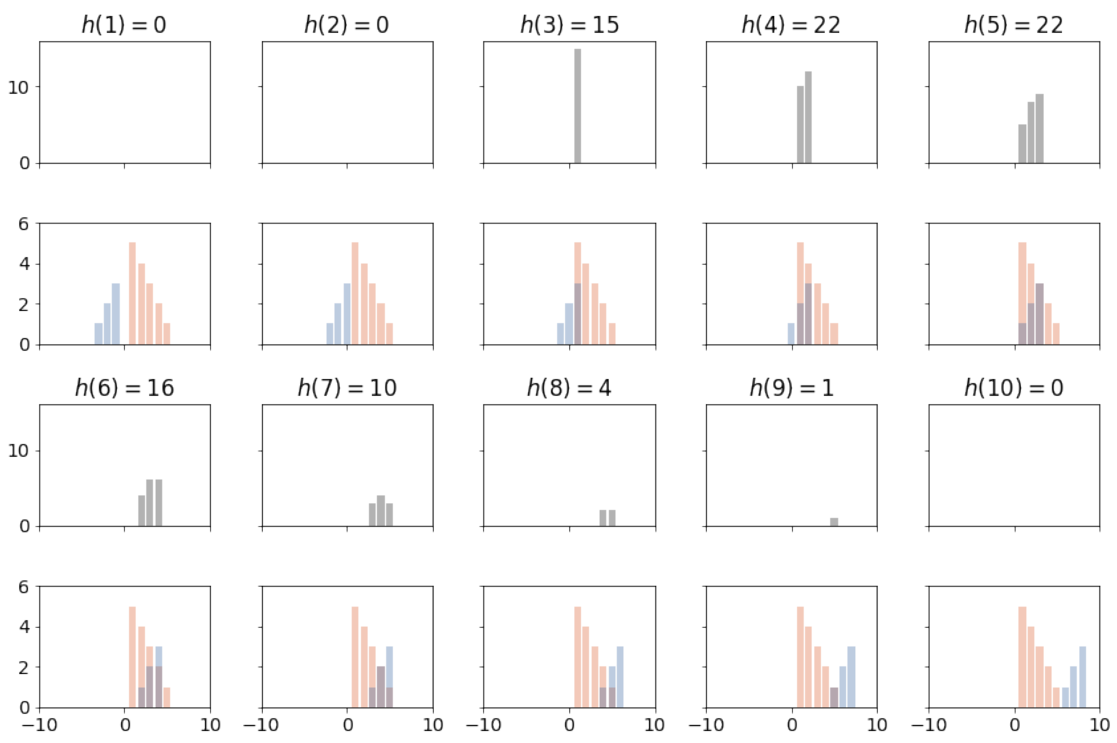
The convolution is commutative, therefore an equivalent definition is:

$$(f \otimes g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \tag{11}$$

[15]:
```python
# convolution demo for two simple signals
def func(x, a, b):
    return (b-x) * ((x>=a) & (x<=b))

def f(x):
    return func(x, 1, 6)
    y = 6 - x
    return y * ((x>=1.) & (x<=6.))

def g(x):
    return func(x, 2, 5)
    y = 5 - x
    return y * ((x>=2.) & (x<=5.))

def plot_func(x, func, ax, color, plot_kw=dict(lw=2, alpha=0.7)):
    f = func(x)
    ax.bar(x, f, color=color, alpha=0.3)

x = np.linspace(-10, 10, 21)
dx = x[1]-x[0]
fig, axes = plt.subplots(1, 4, figsize=(12, 3), sharex='all', sharey='all')
# show signal
ax = axes[0]
ax.set_title(r'signal $f(x)$')
ax.plot(x, f(x), color=colors[0])
plot_func(x, f, ax, colors[0])
# show filter
ax = axes[1]
ax.set_title(r'filter $g(x)$')
ax.plot(x, g(x), color=colors[1])
plot_func(x, g, ax, colors[1])
# reflect g
ax = axes[2]
ax.set_title(r'reflect $g$')
plot_func(x, f, ax, colors[0])
plot_func(x, lambda x: g(-x), ax, colors[1])
ax.legend([r'$f(x)$', r'$g(-x)$'], fontsize=10)
# reflect and shift g
ax = axes[3]
y = -3
ax.set_title(r'reflect and shift $g(y-x)$')
plot_func(x, f, ax, colors[0])
plot_func(x, lambda x: g(y-x), ax, colors[1])
ax.legend([r'$f(x)$', r'$g(y-x)$'], fontsize=10);
fig.tight_layout()
```

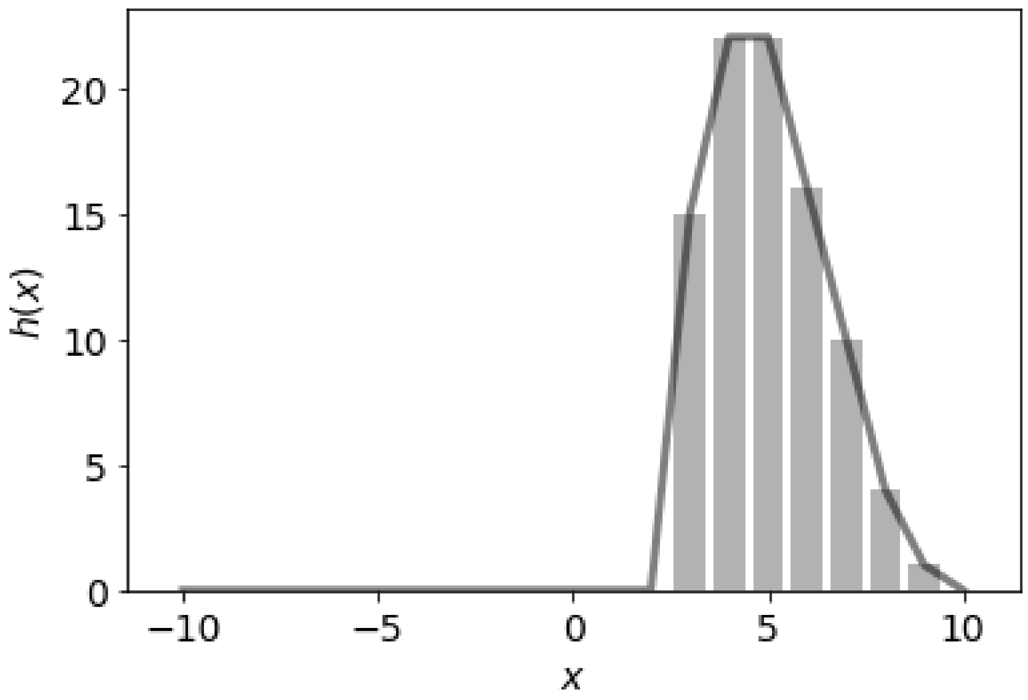


[16]:
```
# convolution
yvals = np.arange(1, 11)
h = []
fig, ax = plt.subplots(4, 5, figsize=(12, 8), sharex='all', sharey='row')
for a, b, y in zip(np.append(ax[0], ax[2]), np.append(ax[1], ax[3]), yvals):
    h.append(f(x) @ g(y-x) * dx)
    a.set_title(r'$h({1})={0:.0f}$'.format(h[-1], y))
    plot_func(x, lambda x, y=y: f(x)*g(y-x), a, colors[3])
    plot_func(x, f, b, colors[0])
    plot_func(x, lambda x, y=y: g(y-x), b, colors[1])
for a in (ax[0,0], ax[2,0]):
    a.set_ylim(0, 16)
for a in (ax[1,0], ax[3,0]):
    a.set_ylim(0, 6)
a.set_xlim(-10, 10.)
fig.tight_layout()
```



The resulting convolution h is:

```
[17]: h = np.array([f(x)@g(y-x) for y in x])
      fig, ax = plt.subplots()
      ax.plot(x, h, color=colors[3], lw=3, alpha=0.5)
      ax.bar(x, h, color=colors[3], alpha=0.3)
      ax.set_xlabel(r'$x$')
      ax.set_ylabel(r'$h(x)$');
```



Properties of the convolution

The convolution is commutative, distributive and associative:

$$
\begin{aligned}
f \otimes g &= g \otimes f \\
f \otimes (g + h) &= f \otimes g + f \otimes h \\
f \otimes (g \otimes h) &= (f \otimes g) \otimes h
\end{aligned} \tag{12}
$$

In NumPy, a convolution can be computed by the function `np.convolve`. For comparison, let us implement a convolution from scratch:

[18]:
```
def convolve(f, g):
    """Returns full convolution.

    Parameters
    ----------
    f : np.ndarray
        The first argument is the signal.

    g : np.ndarray
        The second argument is the filter. It is assumed that
        the size of the signal is at least the size of the
        filter.
    Returns
    -------
    Full convolution stored in an array of size 'len(f)+len(g)-1'
    """
    N, M = len(f), len(g)
    assert N >= M

    # reverse filter
    g = g[::-1]

    # pad zeros to signal
    f = np.pad(f, M-1)

    return np.array([f[i:i+M] @ g for i in range(N+M-1)])
```

[19]:
```
f = np.array([0, 1, 0, 0, 0])
g = np.array([1, 2, 3])

h = np.convolve(f, g)
print(h)
print(convolve(f, g))
print(np.allclose(h, convolve(f, g)))
```

```
[0 1 2 3 0 0 0]
[0 1 2 3 0 0 0]
True
```

NumPy's `np.pad(f, m)` command adds `m` zeros at the beginning and end of the array `f`:

[20]:
```
m = 4
print('[' + ' '.join(np.pad(['f[0]', 'f[1]', '...', 'f[n-1]'], m)) + ']')
```

```
[0 0 0 0 f[0] f[1] ... f[n-1] 0 0 0 0]
```

If necessary, the result of a convolution can be truncated to the size of the original arrays. The corresponding NumPy command is:

[21]:
```
f = np.array([0, 1, 0, 0, 0])
g = np.array([1, 2, 3])
print(np.convolve(f, g, 'full')) # full convolution
print(np.convolve(f, g))         # full convolution is returned by default
print(np.convolve(f, g, 'same')) # same size as f
```

```
[0 1 2 3 0 0 0]
[0 1 2 3 0 0 0]
[1 2 3 0 0]
```

The convolution theorem

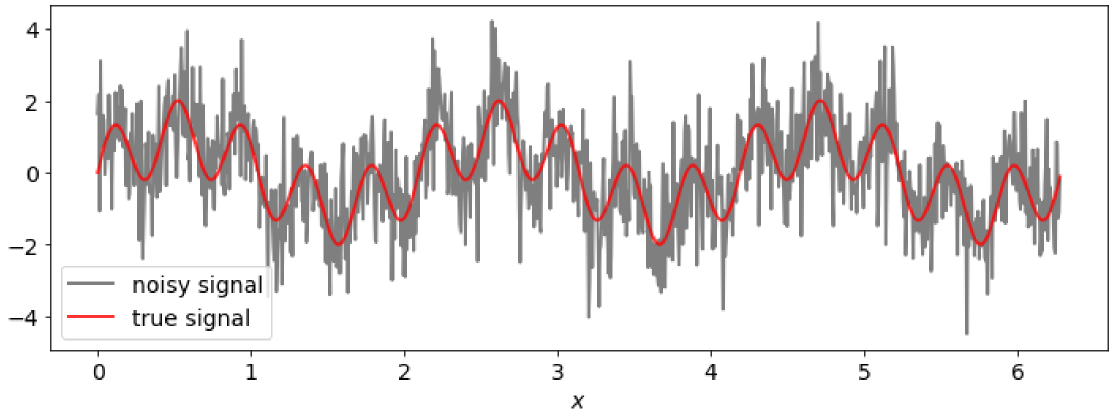
The convolution theorem states that the Fourier transform of a convolution of two signals is the pointwise product of their Fourier transforms:

$$\mathcal{F}\{f \otimes g\} = \mathcal{F}\{f\} \odot \mathcal{F}\{g\} \tag{13}$$

where $\mathcal{F}$ denotes the Fourier transform of the functions and $\odot$ is the element-wise product (also known as Hadamard product). In other words, convolution in one domain (e.g. time or spatial domain) equals element-wise multiplication in the other domain (e.g. frequency domain).

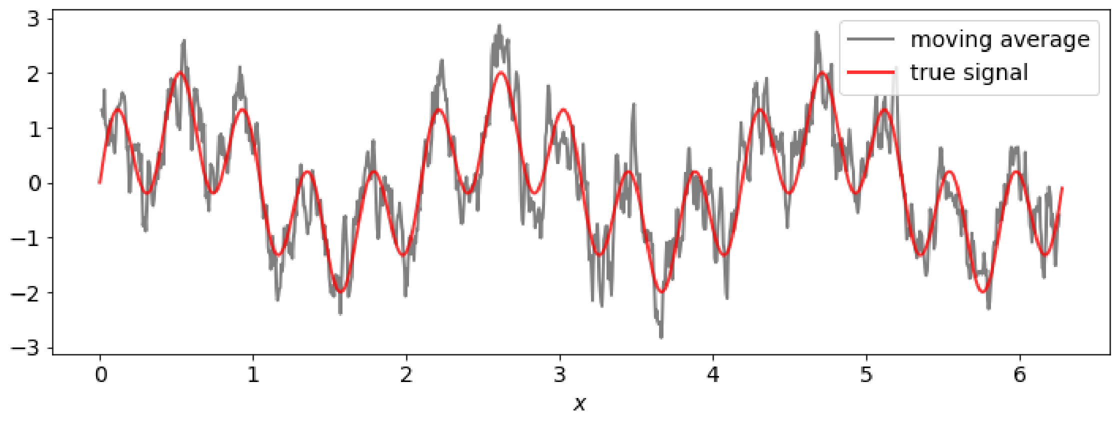
Application: denoising by a sliding or moving average

[22]:
```
# true signal
N = 2**10
x = np.linspace(0., 2*np.pi, N, endpoint=False)
freqs = [3, 15]
signal = np.sum(np.sin(np.multiply.outer(x, freqs)), axis=1)

# add some noise with chosen signal-to-noise ratio (SNR)
snr = 1
sigma = np.sqrt(np.var(signal)/snr)
noise = np.random.standard_normal(signal.shape) * sigma

# observed: signal + noise
f = signal + noise

plot_kw = dict(color='k', alpha=0.5, lw=2)
fig, ax = plt.subplots(figsize=(12, 4))
ax.plot(x, f, **plot_kw, label='noisy signal')
ax.plot(x, signal, color='r', lw=2, alpha=0.8, label='true signal')
ax.set_xlabel(r'$x$')
ax.legend();
```

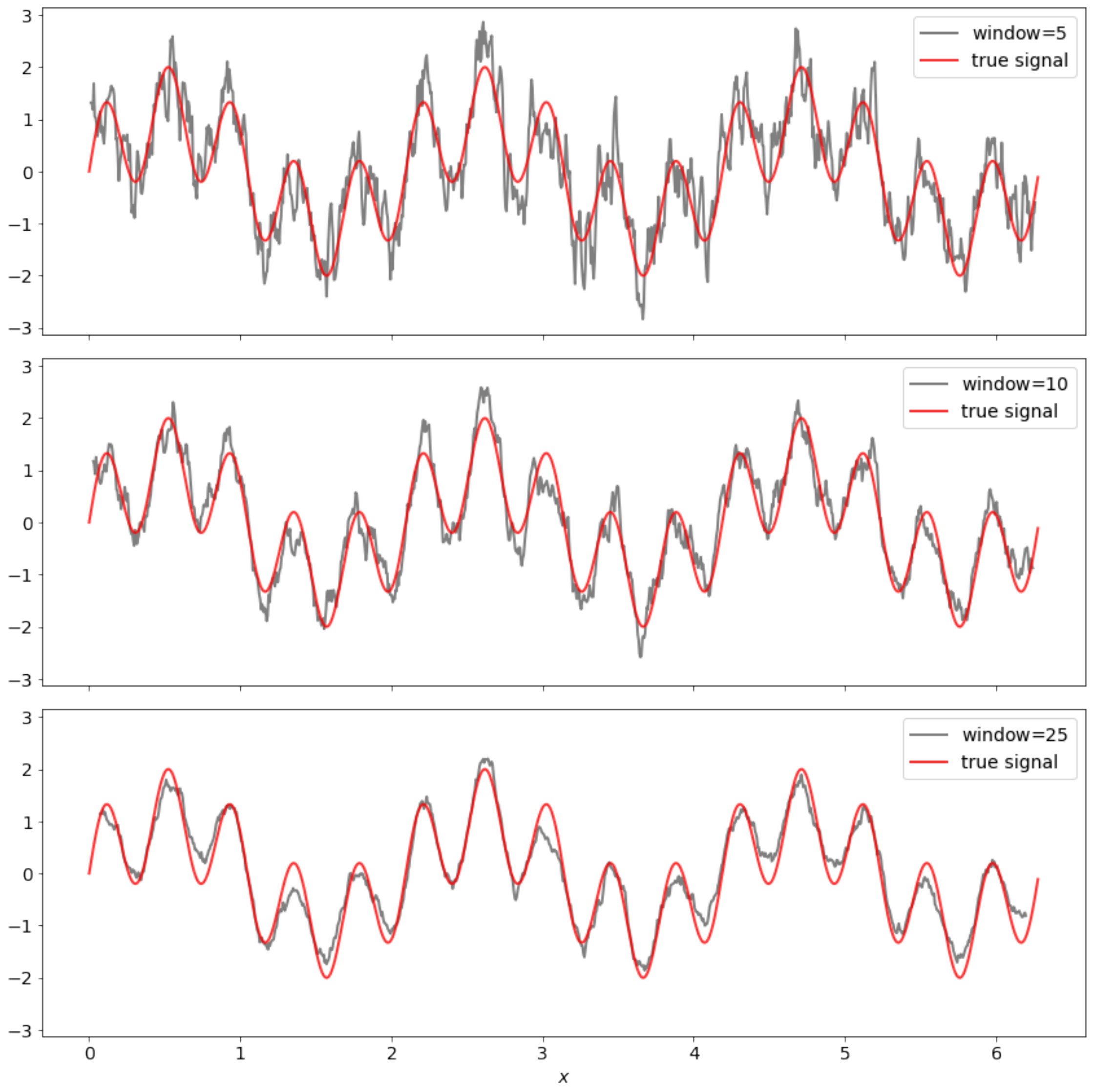


We want to recover the true signal from the noisy observation. A simple approach is to compute a *moving* average, because the noise averages out:

[23]:
```
window = 5
x_average = [np.mean(x[i-window:i]) for i in range(window, N)]
f_average = [np.mean(f[i-window:i]) for i in range(window, N)]

fig, ax = plt.subplots(figsize=(12, 4))
ax.plot(x_average, f_average, **plot_kw, label='moving average')
ax.plot(x, signal, color='r', lw=2, alpha=0.8, label='true signal')
ax.set_xlabel(r'$x$')
ax.legend();
```



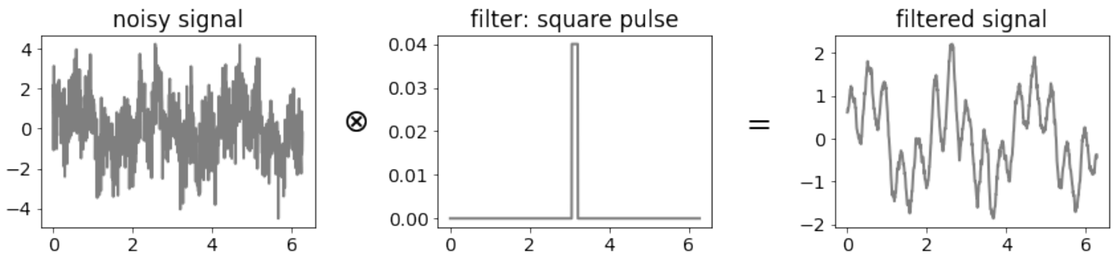
[24]:
```
# let's define a convenience function
def moving_average(x, window=5):
    return np.array([np.mean(x[i-window:i]) for i in range(window, N)])

fig, ax = plt.subplots(3, 1, figsize=(12, 12), sharex='all', sharey='all')
for i, window in enumerate([5, 10, 25]):
    ax[i].plot(moving_average(x, window), moving_average(f, window),
               **plot_kw, label=r'window=${}$'.format(window))
    ax[i].plot(x, signal, color='r', lw=2, alpha=0.8, label='true signal')
    ax[i].legend()
ax[i].set_xlabel(r'$x$')
fig.tight_layout()
```



The sliding average can be interpreted as a convolution with a rectangular pulse of size w (where w is the window used in the averaging):

[25]:
```
g = np.zeros_like(f)
g[:window//2] = 1/window
g[-window//2:] = 1/window
g = np.fft.ifftshift(g)

fig, ax = plt.subplots(1, 3, figsize=(12, 3), sharex='all')
ax[0].set_title('noisy signal')
ax[0].plot(x, f, **plot_kw)
ax[0].annotate(r'$\otimes$', xy=(1.1, .5), fontsize=24, xycoords='axes fraction')
ax[1].set_title('filter: square pulse')
```

```
ax[1].plot(x, g, **plot_kw)
ax[1].annotate(r'$=$', xy=(1.1, .5), fontsize=24, xycoords='axes fraction')
ax[2].set_title('filtered signal')
ax[2].plot(x, np.convolve(f, g, mode='same'), **plot_kw)
fig.tight_layout()
```



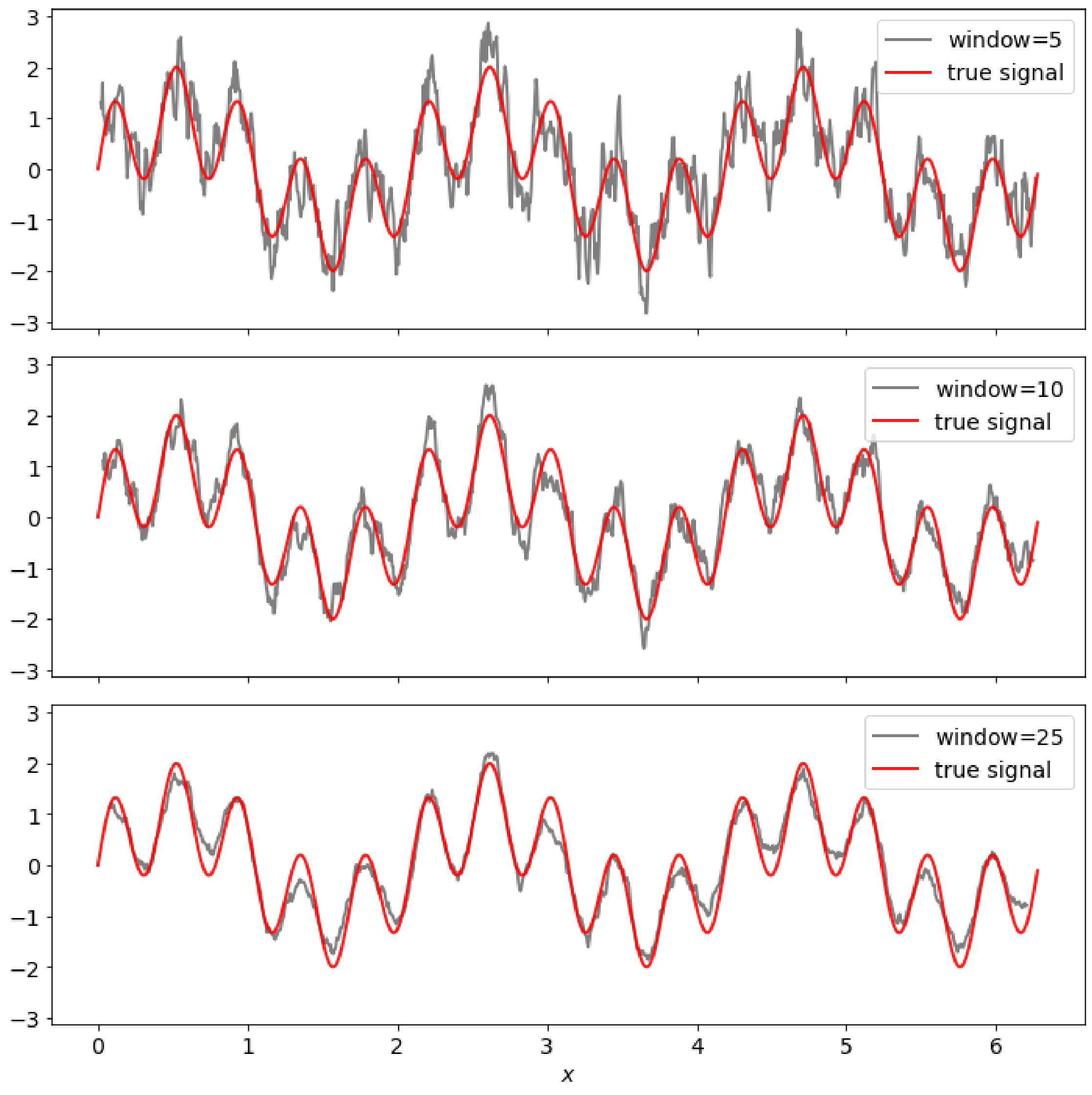
So we can use the convolution theorem to implement the moving average. In NumPy, this boils down to

```
filtered_signal = np.fft.ifft(np.fft.fft(signal) * np.fft.fft(filter))
```

Notice that `signal` and `filter` must have the same size. If this is not the case `filter` must be padded with zeros so as to match the size of `signal`.

[26]:
```
def filter_signal(signal, filter):
    if filter.size < signal.size:
        filter = np.pad(signal, signal.size//2)
    return np.real(np.fft.ifft(np.fft.fft(signal) * np.fft.fft(filter)))

def rectangular_pulse(size, window_size):
    """Generate a rectangular pulse of size `window_size`. """
    rect = np.zeros(size)
    # since signals are assumed to start at zero we have to
    # set the first and last entries to 1/window
    rect[:window//2] = 1/window
    rect[-window//2:] = 1/window
    return np.fft.ifftshift(rect)

def moving_average_fft(f, window=5):
    g = rectangular_pulse(f.size, window)
    # need to more x=0 to the start
    g = np.fft.fftshift(g)
    return filter_signal(f, g)[window//2:-window//2]
```

[27]:
```
fig, ax = plt.subplots(3, 1, figsize=(10, 10), sharex='all', sharey='all')
for i, window in enumerate([5, 10, 25]):
    ax[i].plot(moving_average_fft(x, window),
               moving_average_fft(f, window),
                **plot_kw, label=r'window=${}$'.format(window))
    ax[i].plot(x, signal, color='r', lw=2, alpha=0.9, label='true signal')
    ax[i].legend()
ax[i].set_xlabel(r'$x$')
fig.tight_layout()
```



NumPy functions for convolution in 1 dimension

There are various options for computing the convolution of two signals in NumPy:

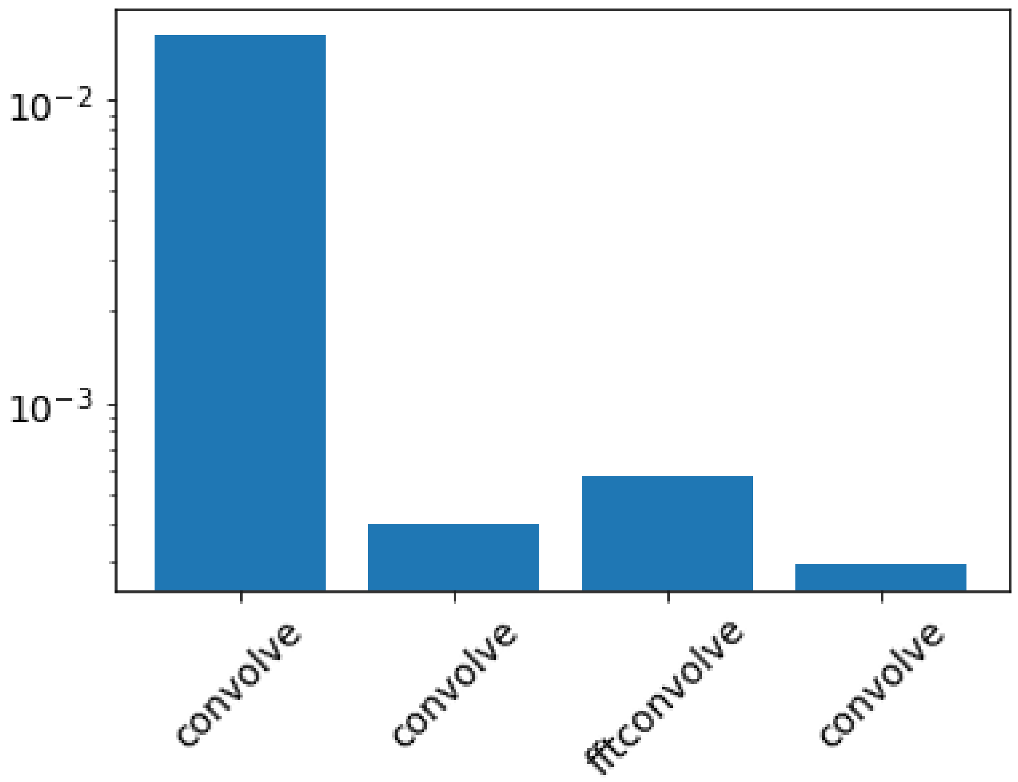
1. **FFT**: `np.fft.ifft(np.fft.fft(f) * np.fft.fft(g))`. Note that `f` and `g` must be padded with zeros so that they have the same size.
2. **NumPy**: For convolving arrays also the command `np.convolve(f, g)` can be used. `f` and

g do not need to have the same size. `np.convolve(f, g, 'full')` returns the convolution of vectors `f` and `g`. Calling `np.convolve(f, g, 'same')` will return the central part of the convolution having the same size as `f`.

3. **SciPy**: The submodules `scipy.signal` offers two implementations of convolutions of 1D signals: `scipy.signal.convolve` and `scipy.signal.fftconvolve`

[28]:
```
# comparison of convolution functions
import time
import scipy.signal

funcs = [convolve,
         np.convolve,
         scipy.signal.convolve,
         scipy.signal.fftconvolve]

n, m = 2**13, 2**6

f = np.random.standard_normal(n)
g = np.random.standard_normal(m)

times = []
funcs = [convolve,
         np.convolve,
         scipy.signal.fftconvolve,
         scipy.signal.convolve]

h = []
for func in funcs:
    start = time.process_time()
    h.append(func(f, g))
    times.append(time.process_time() - start)

for i, func in enumerate(funcs):
    print(func.__name__, np.allclose(h[0], h[i]))

plt.close('all')
fig, ax = plt.subplots()
ax.bar(np.arange(len(funcs)), times)
ax.semilogy()
ax.set_xticklabels([func.__name__ for func in funcs], rotation=45)
ax.set_xticks(np.arange(len(funcs)));
```

```
convolve True
convolve True
fftconvolve True
convolve True
```



Smoothing spatial filters

Convolution of a 2D filter with a discrete unit pulse (2D delta peak)

[29]:
```python
# pulse
n = 2
f = np.zeros((2*n+1, 2*n+1), dtype=int)
f[n, n] = 1

# filter
w = np.arange(1, 10, dtype=int).reshape(3, 3)

print('pulse:\n{}\n\nfilter:\n{}'.format(f, w))
```

```
pulse:
[[0 0 0 0 0]
 [0 0 0 0 0]
 [0 0 1 0 0]
 [0 0 0 0 0]
 [0 0 0 0 0]]

filter:
```

```
[[1 2 3]
 [4 5 6]
 [7 8 9]]
```

[30]:
```
# padding
f_padded = np.pad(f, len(w)-1)

print('padded filter:\n{}'.format(f_padded))
```

```
padded filter:
[[0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0]
 [0 0 0 0 1 0 0 0 0]
 [0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0]]
```

[31]:
```
# reversed filter
w_reversed = w[::-1, ::-1]

print('reversed filter:\n{}'.format(w_reversed))
```

```
reversed filter:
[[9 8 7]
 [6 5 4]
 [3 2 1]]
```

[32]:
```
for i in range(f_padded.shape[0]-len(w)+1):
    for j in range(f_padded.shape[1]-len(w)+1):
        h = np.sum(f_padded[i:i+len(w), j:j+len(w)] * w_reversed)
        print(h, end='    ')
    print('\n')
```

```
0    0    0    0    0    0    0

0    0    0    0    0    0    0

0    0    1    2    3    0    0

0    0    4    5    6    0    0

0    0    7    8    9    0    0

0    0    0    0    0    0    0
```

```
0    0    0    0    0    0    0
```

[33]:
```
def convolve2d(f, g):
    """Full convolution of 2D signals. """
    n, m = len(f), len(g)
    # reverse filter
    g = g[::-1, ::-1]

    # pad zeros to signal
    f = np.pad(f, m-1)

    h = [np.sum(f[i:i+m, j:j+m] * g)
         for i in range(n+m-1) for j in range(n+m-1)]

    return np.array(h).reshape(n+m-1, n+m-1)
```

[34]:
```
# apply to example
print('pulse:\n' + '='*14)
print(f)

print('\nfilter:\n' + '='*10)
print(w)

print('\nfiltered pulse:\n' + '='*18)
print(convolve2d(f, w))
```

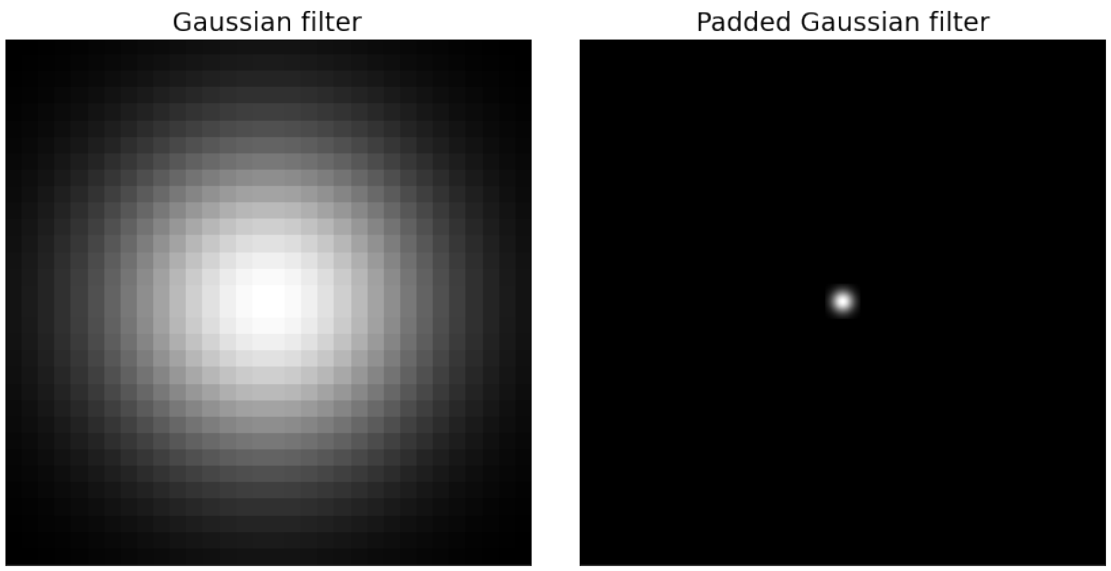
```
pulse:
==============
[[0 0 0 0 0]
 [0 0 0 0 0]
 [0 0 1 0 0]
 [0 0 0 0 0]
 [0 0 0 0 0]]

filter:
==========
[[1 2 3]
 [4 5 6]
 [7 8 9]]

filtered pulse:
==================
[[0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0]
 [0 0 1 2 3 0 0]
 [0 0 4 5 6 0 0]
 [0 0 7 8 9 0 0]
 [0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0]]
```

[35]:
```
def circle(f, radius, center=None, tolerance=1, angle_range=None):
    if center is None:
        center = [f.shape[0]//2, f.shape[1]//2]
    i, j = np.indices(f.shape)
    x, y = i-center[0], j-center[1]
    r = np.sqrt(x**2 + y**2)
    mask = np.fabs(r - radius) <= tolerance
    if angle_range is not None:
        angles = np.arctan2(y, x)
        mask &= (angles >= angle_range[0]) & (angles <= angle_range[1])
    f[mask] = 1

def make_smiley(npix=256):
    f = np.zeros((npix, npix))
    circle(f, 100, tolerance=5)
    circle(f, 5, [npix*3//8, npix*3//8], tolerance=5)
    circle(f, 5, [npix*3//8, npix*5//8], tolerance=5)
    circle(f, 5, [npix//2, npix//2], tolerance=5)
    circle(f, 65, [npix//2, npix//2], tolerance=5,
           angle_range=(np.deg2rad(-55), np.deg2rad(55)))
    return f

# show image
f = make_smiley()
subplot_kw = dict(xticks=[], yticks=[])
fig, ax = plt.subplots(figsize=(7, 7), subplot_kw=subplot_kw)
ax.imshow(f, cmap='gray');
```

```
[36]: x = np.linspace(-5., 5., 2**5)
      X, Y = np.meshgrid(x, x)
      # Gaussian filter
      g = gaussian = np.exp(-0.1 * (X**2 + Y**2))
      # Disk-shaped filter
      if not True:
          g[...] = 0.
          g[X**2+Y**2 < 10] = 1.
      h = scipy.signal.convolve2d(f, g)
      h2 = convolve2d(f, g)
      print('Local version agrees with scipy?', np.allclose(h, h2))
      g_pad = np.pad(g, len(f)-len(g))
      plt.rc('font', size=16)
      fig, ax = plt.subplots(1, 2, figsize=(12, 6), subplot_kw=subplot_kw)
      ax[0].set_title('Gaussian filter')
```

```
ax[0].imshow(g, cmap='gray');
ax[1].set_title('Padded Gaussian filter')
ax[1].imshow(g_pad, cmap='gray');
fig.tight_layout()
```

```
Local version agrees with scipy? True
```



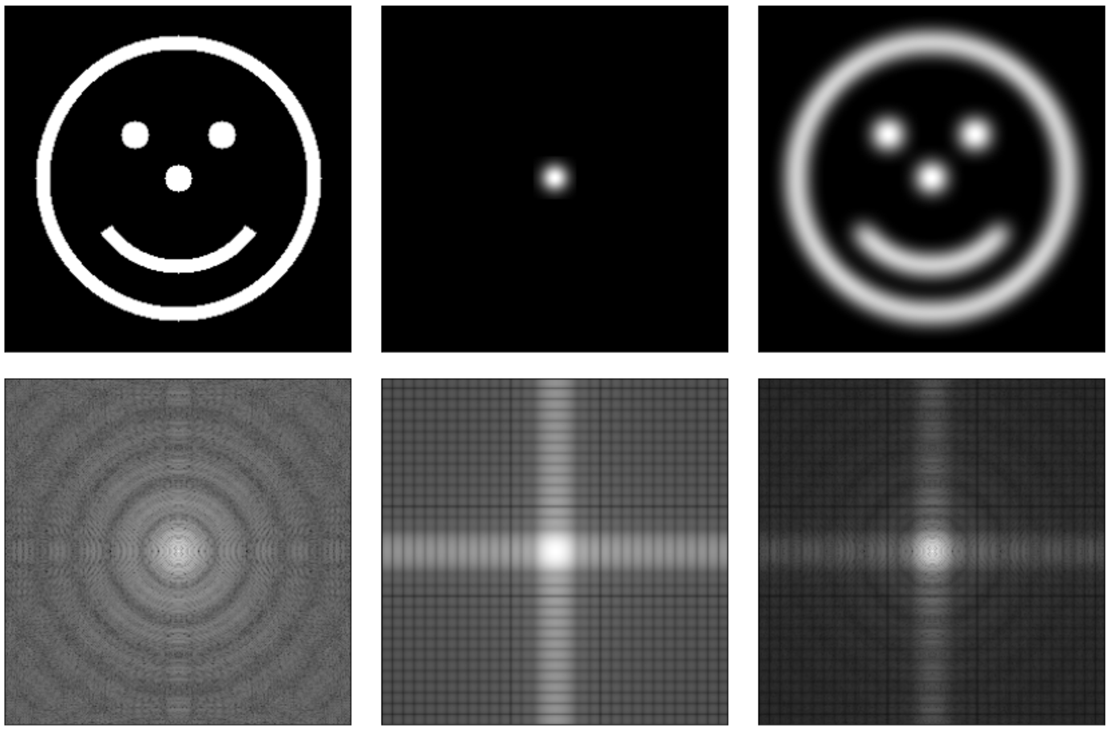
[37]:
```
print(f.shape, g.shape, h.shape)
fig, ax = plt.subplots(figsize=(7, 7), subplot_kw=subplot_kw)
ax.imshow(h, cmap='gray');
```

```
(256, 256) (32, 32) (287, 287)
```

We can also use the FFT trick in 2D:

[38]:
```
def fftconvolve(f, g):
    n, m = len(f), len(g)
    g = np.fft.ifftshift(np.pad(g, (n-m)//2))
    F = np.fft.fftn(f)
    G = np.fft.fftn(g)
    return np.real(np.fft.ifftn(F * G))

fig, ax = plt.subplots(1, 2, figsize=(12, 6), subplot_kw=subplot_kw)
ax[0].imshow(scipy.signal.convolve2d(f, g, mode='same'), cmap='gray')
ax[1].imshow(fftconvolve(f, g), cmap='gray');
fig.tight_layout();
```

[39]:
```
F = np.fft.fftshift(np.fft.fftn(f))
G = np.fft.fftshift(np.fft.fftn(
    np.fft.ifftshift(np.pad(g, (len(f)-len(g))//2))
))

gamma = 0.1
fig, ax = plt.subplots(2, 3, figsize=(12, 8), subplot_kw=subplot_kw)
ax[0, 0].imshow(f, cmap='gray')
ax[0, 1].imshow(np.pad(g, (len(f)-len(g))//2), cmap='gray');
ax[0, 2].imshow(fftconvolve(f, g), cmap='gray');
ax[1, 0].imshow(np.abs(F)**gamma, cmap='gray')
ax[1, 1].imshow(np.abs(G)**gamma, cmap='gray');
ax[1, 2].imshow(np.abs(F * G)**gamma, cmap='gray');
fig.tight_layout();
```



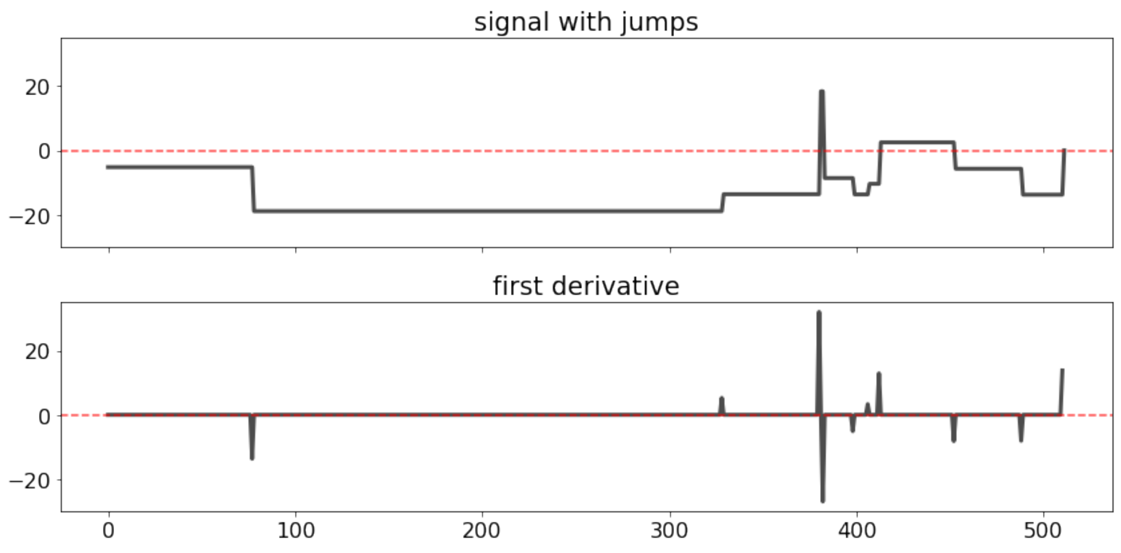
Sharpening spatial filters

Spatial filters can also be used to *sharpen* an image rather than blurring it. Derivatives can be used to detect edges. The first and second derivative of a 1D signal can be approximated by finite differences:

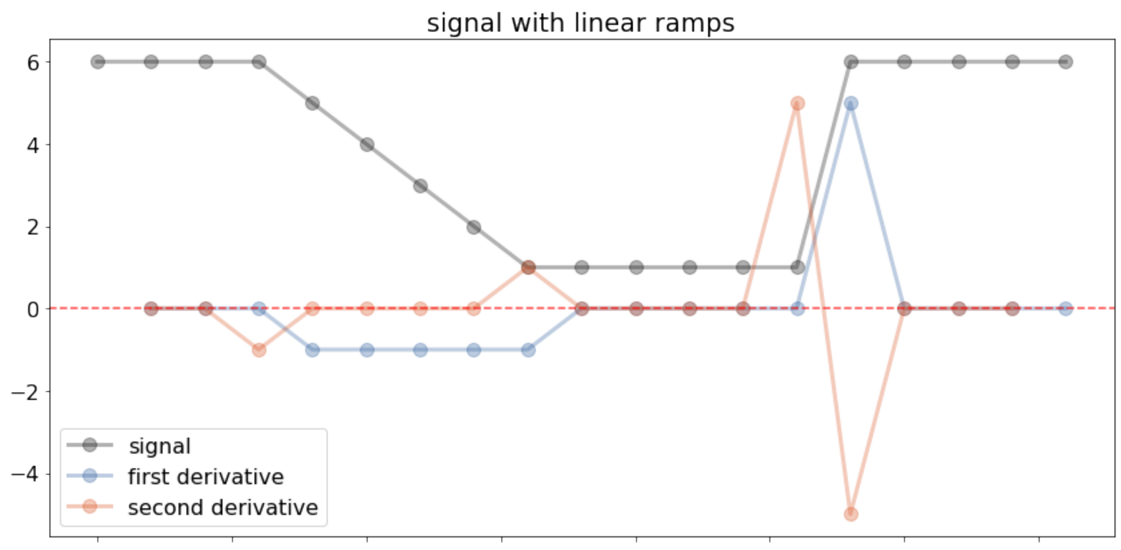
$$\frac{d}{dx}f(x) \to f[i+1] - f[i]$$
$$\frac{d^2}{dx^2}f(x) \to f[i+1] - 2 * f[i] + f[i-1]$$

[40]:
```
# 1D signal with change points
N = 2**9
# function levels and number of change points
f_min, f_max, n = -20, 20, 10
levels = np.random.uniform(f_min, f_max, n)
change_points = np.append(
    np.sort(np.random.randint(0, N, n)), N-1
)
f = np.zeros(N)
for start, end, level in zip(change_points, change_points[1:], levels):
    f[start:end] = level
```

```
plot_kw = dict(color='k', lw=3, alpha=0.7)
fig, axes = plt.subplots(2, 1, figsize=(12, 6), sharex='all', sharey='all')
axes[0].set_title('signal with jumps')
axes[0].plot(f, **plot_kw)
axes[1].set_title('first derivative')
axes[1].plot(f[1:]-f[:-1], **plot_kw)
for ax in axes:
    ax.axhline(0., ls='--', color='r', alpha=0.7)
fig.tight_layout();
```



[41]:
```
# signal with linear ramps

f = np.array([6, 6, 6, 6, 5, 4, 3, 2, 1, 1, 1, 1, 1, 1, 6, 6, 6, 6, 6])
x = np.arange(len(f))

plot_kw = dict(lw=3, alpha=0.3, marker='o', markersize=10)
fig, ax = plt.subplots(figsize=(12, 6), sharex='all', sharey='all')
ax.set_title('signal with linear ramps')
ax.plot(x, f, color=colors[3], label='signal', **plot_kw)
ax.plot(x[1:], f[1:]-f[:-1], label='first derivative', color=colors[1],␣
 ↪**plot_kw)
ax.plot(x[1:-1], f[2:] - 2*f[1:-1] + f[:-2], label='second derivative',
        color=colors[0], **plot_kw)
ax.axhline(0., ls='--', color='r', alpha=0.7)
ax.set_xticklabels([])
ax.legend()
fig.tight_layout();
```



In two-dimensions the finite difference version of the *Laplace* operator:

$$\Delta f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

Is obtained from

$$\begin{aligned}\frac{\partial^2}{\partial x^2} f(x,y) &\rightarrow f[i+1,j] - 2 * f[i,j] + f[i-1,j] \\ \frac{\partial^2}{\partial y^2} f(x,y) &\rightarrow f[i,j+1] - 2 * f[i,j] + f[i,j-1]\end{aligned}$$

resulting in

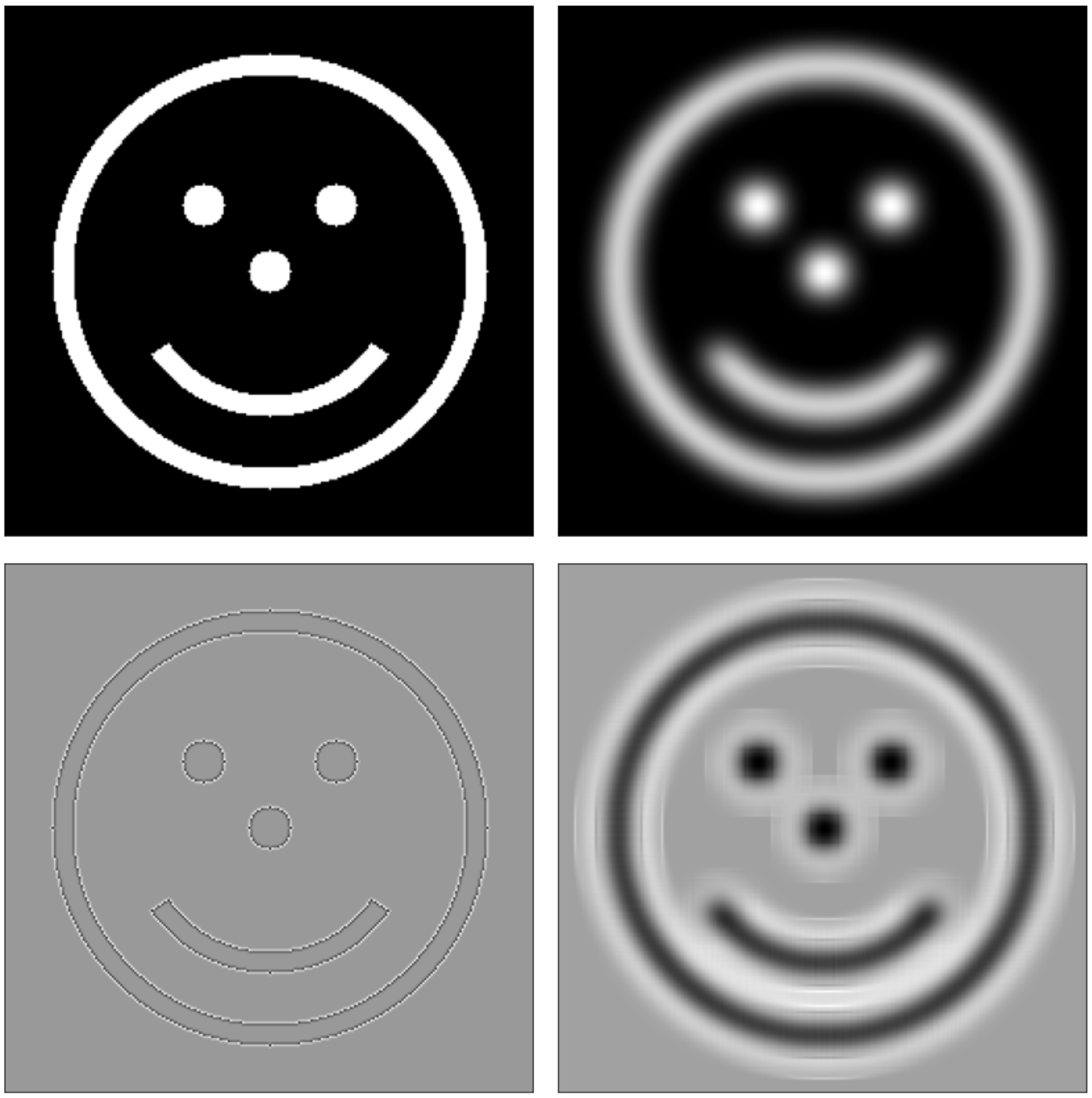
$$\Delta f(x,y) \rightarrow f[i,j+1] + f[i,j-1] + f[i+1,j] + f[i-1,j] - 4 * f[i,j]$$

This is a convolution with the spatial filter:

$$\Delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{14}$$

[42]:
```
g = np.array([[0, 1, 0],
              [1, -4, 1],
              [0, 1, 0]])
f = make_smiley()
h = scipy.signal.convolve2d(f, gaussian, mode='same')
f_sharp = scipy.signal.convolve2d(f, g, mode='same')
```

```
h_sharp = scipy.signal.convolve2d(h, g, mode='same')

fig, axes = plt.subplots(2, 2, figsize=(12, 12), subplot_kw=subplot_kw)
axes[0, 0].imshow(f, cmap='gray')
axes[0, 1].imshow(h, cmap='gray')
axes[1, 0].imshow(f_sharp, cmap='gray')
axes[1, 1].imshow(h_sharp, cmap='gray')
fig.tight_layout()
```



[43]:
```
g = np.array([[1, 1, 1],
              [1, -8, 1],
              [1, 1, 1]])
f = make_smiley()
```

```python
h = scipy.signal.convolve2d(f, gaussian, mode='same')
f_sharp = scipy.signal.convolve2d(f, g, mode='same')
h_sharp = scipy.signal.convolve2d(h, g, mode='same')

fig, axes = plt.subplots(2, 2, figsize=(12, 12), subplot_kw=subplot_kw)
axes[0, 0].imshow(f, cmap='gray')
axes[0, 1].imshow(h, cmap='gray')
axes[1, 0].imshow(f_sharp, cmap='gray')
axes[1, 1].imshow(h_sharp, cmap='gray')
fig.tight_layout()
```